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THE CROSSED COPRODUCT THEOREM AND GALOIS COHOMOLOGY

BY

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Dedicated to the memory of Prof. S. A. Amitsur

ABSTRACT

We give a cohomological interpretation of the Brauer group of a coalgebra in terms of Galois coextensions and Galois cohomology. There is a crossed coproduct structure theorem, and the co-version of the classical splitting theorem holds for the Brauer group of an irreducible coreflexive coalgebra but it does not hold in general.

Introduction

The crossed product theorem in the classical theory of the Brauer group relates the subgroup of the Brauer group split by a given extension to the second (Galois) cohomology group. This result provides us with an effective method to calculate explicitly in the Brauer group and it reduces problems for algebras to problems for groups.

In [15], we introduced the Brauer group of a cocommutative coalgebra; the classical theory dualized well enough but the 'duality' did not follow in a straightforward way because of some particular coalgebra effects that produced themselves. For example, the Brauer group need not be a torsion group. On the

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other hand, some cohomology theories do exist, e.g. Sweedler cohomology [12] or Doi cohomology [4]; some crossed coproducts for Hopf algebras have already been investigated in [4, 6]. It seems to be natural to try to extend this theory for coalgebras, looking for a cohomological description of the Brauer group of a cocommutative coalgebra. This is the main aim of this paper. For a finite dimensional commutative Hopf algebra we prove that Doi cohomology and Sweedler cohomology coincide (Lemma 3.1). On the other hand, Sweedler cohomology and Galois cohomology coincide (with Amitsur cohomology cf. [1, 12]) for Hopf algebras which are dual to a group Hopf algebra. This makes it possible to extend the classical theory of crossed products to the case of Galois coextensions with respect to finite groups. The comodule theory of coalgebras may be considered 'easier ' than its algebras equivalents because of the in-built finiteness aspects of coalgebras; nevertheless in Brauer group theory these aspects do not interfere at all. The extension of the classical theory of crossed products to the case of Galois coextensions w.r.t. finite groups does contain the relation between the crossed coproduct structure and the existence of maximal cocommutative quotient coalgebras (Prop. 3.3, Prop. 3.6). In fact, Proposition 3.6 replaces a classical result due to M. Auslander and O. Goldman, stating that an Azumaya algebra A over a commutative ring S split by a finitely generated projective extension T of S is equivalent to an Azumaya algebra containing T/S as a maximal commutative subring. In a similar way, Proposition 3.7 will replace the classical Skolem–Noether theorem. Finally we arrive at the main result, Theorem 3.8, that replaces the classical crossed product theorem describing the split part of the Brauer group as a second Galois cohomology group. That is, for a finite group G and a cocommutative coextension C/R such that C/R is kG-Galois, we obtain $\mathrm{Br}(C/R) = H^2(C, kG^*)$. This interpretation gives a completely cohomological description of the Brauer group of an irreducible coreflexive coalgebra (cf. Th.3.10 and Cor.3.11), and gives an affirmative answer to the question proposed in [15, p. 568]. Precisely, if R is an irreducible coreflexive coalgebra, then any Azumaya R-coalgebra D is split by a Galois coextension C of R with respect to a finite group G. This splitting theorem fails in general for not necessarily coreflexive coalgebras where the complete calculation may be possible but would reflect some of the usual topological features of 'taking the dual'.

1. Preliminaries

Throughout k is a fixed field. All coalgebras, algebras, vector spaces and unadorned $\otimes$, Hom, etc. are over k. C, D and E always denote coalgebras. We refer to [13] for full detail on coalgebras and comodules. We adopt the usual sigma notation for the comultiplication of a coalgebra, and adopt the following sigma notation for a (left) C-comodule structure ρ_X of X:

$$\rho(x) = \sum x_{\langle -1\rangle} \otimes x_{\langle 0\rangle}.$$

Let $\mathbf{M}^C$ (or ${}^C\mathbf{M}$) denote the category of right (or left) C-comodules. If $\alpha\colon C \longrightarrow D$ is a coalgebra map, then any left C-comodule X may be treated as a left D-comodule in a natural way:

$$(1) \qquad (\alpha \otimes 1)\rho\colon X \longrightarrow C \otimes X \longrightarrow D \otimes X.$$

In particular, C is a left (or right) D-comodule. A $(C-D)$-bicomodule is a left C-comodule and a right D-comodule X, denoted by ${}_CX_D$, such that the C-comodule structure map $\rho_C\colon X \longrightarrow C \otimes X$ is right D-colinear (or a D-comodule map).

For a right C-comodule M and a left C-comodule N, the cotensor product $M \Box_C N$ is the kernel of

$$\rho_M \otimes 1 - 1 \otimes \rho_N\colon M \otimes N \rightrightarrows M \otimes C \otimes N.$$

The functors $M\Box_C -$ and $-\Box_C N$ are left exact and preserve direct sums. If ${}_CX_D$ and ${}_DY_E$ are bicomodules, then $X\Box_D Y$ is a $(C-E)$-bicomodule with comodule structures induced by those of X and Y.

A comodule X_C is quasi-finite if $\mathrm{Com}_{-C}(Y, X)$ is finite dimensional for any finite dimensional comodule Y_C. Similarly, one may define the quasi-finite right comodules over C. A comodule X_C is finitely cogenerated if it is isomorphic to a subcomodule of $W \otimes C$ for some finite dimensional space W. A finitely cogenerated comodule is quasi-finite. But the converse is not true. A comodule $X \in \mathbf{M}^C$ is said to be a cogenerator if for any comodule $M \in \mathbf{M}^C$ there is a space W such that $M \hookrightarrow W \otimes X$ as comodules. Let $\alpha\colon C \longrightarrow D$ be a surjective coalgebra map and let ${}_CX$ be a left C-comodule. If X is quasi-finite as a left D-comodule via (1), then ${}_CX$ is quasi-finite.

In fact, for any finite dimensional left C-comodule Y, we have an inclusion of vector spaces:

$$\mathrm{Com}_C(Y, X) \subseteq \mathrm{Com}_D(Y, X).$$

This is because a C-colinear map from Y to X is simultaneously a D-colinear and k-linear map. Consequently $\mathrm{Com}_C(Y,X)$ is finite dimensional.

Now we recall from [14] the definition of the co-hom functor and some of its basic properties. The following result is fundamental:

BASIC LEMMA: *Let ${}_CX_D$ be a bicomodule. Then X_D is quasi-finite if and only if the functor $-\Box_C X$: $\mathbf{M}^C \longrightarrow \mathbf{M}^D$ has a left adjoint functor, denoted by $h_{-D}(X,-)$. That is, for comodules Y_D and W_C,*

$$\mathrm{Com}_{-C}(h_{-D}(X,Y),W) \simeq \mathrm{Com}_{-D}(Y, W\Box_C X). \tag{2}$$

where

$$h_{-D}(X,Y) = \varinjlim_{\mu} \mathrm{Com}_{-D}(Y_\mu, X)^* \simeq \varinjlim_{\mu} (X\Box_D Y_\mu^*)^*$$

is a right C-comodule, $\{Y_\mu\}$ is a directed family of finite dimensional subcomodules of Y_D such that $Y = \bigcup_\mu Y_\mu$. In particular, if $C = k$, $X = D$, then $h_{-D}(D,-)$ is nothing but the forgetful functor U: $\mathbf{M}^D \longrightarrow$ M (here $\mathbf{M}$ is the k-module category) if $C = D$, $X = D$, $h_{-D}(D,-)$ is the identity functor from $\mathbf{M}^D$ to $\mathbf{M}^D$. Let θ denote the canonical D-colinear map $Y \longrightarrow h_{-D}(X,Y)\Box_C X$ which corresponds to the identity map $h_{-D}(X,Y) \longrightarrow h_{-D}(X,Y)$ in (2). Similarly, there is a left version of the basic lemma for the quasi-finite comodule ${}_CX$.

Assume that X_D is a quasi-finite comodule. $e_{-D}(X) = h_{-D}(X,X)$ is a coalgebra, called the co-endomorphism coalgebra of X. The comultiplication of $e_{-D}(X)$ corresponds to $(1\otimes\theta)\theta$: $X \longrightarrow e_{-D}(X)\otimes e_{-D}(X)\otimes X$ in (2) when $C = k$, and the counit of $e_{-D}(X)$ corresponds to the identity map 1_X. X is an $e_{-D}(X) - D$-bicomodule with the left comodule structure map θ, given by the canonical map $X \longrightarrow h_{-D}(X,X)\otimes X$.

Let ${}_CX_D$ be a bicomodule such that X_D is quasi-finite. Then there exists a coalgebra map

$$\lambda\colon e_{-D}(X) \longrightarrow C \tag{3}$$

such that the left C-comodule structure equals $(\lambda\otimes 1)\theta$. Conversely, a coalgebra map λ: $e_{-D}(X) \longrightarrow C$ makes X into a $C-D$-bicomodule. Moreover, the $C-C$-bicomodule structure of $e_{-D}(X)$ through λ coincides with the induced $C-C$-bicomodule structure of $h_{-D}({}_CX_D, {}_CX_D)$. Similarly, if ${}_CX$ is quasi-finite, there exists a coalgebra map λ': $e_{C-}(X) \longrightarrow D$ such that ρ_D: $X \longrightarrow X\otimes D$ is of form $(1\otimes\lambda')\theta'$, where θ': $X \longrightarrow X\otimes e_{C-}(X)$ is the canonical map.

A Morita–Takeuchi (M-T) context $(C, D, {}_CP_D, {}_DQ_C, F, G)$ consists of coalgebras C, D, bicomodules ${}_CP_D, {}_DQ_C$, and bicolinear maps $F: C \longrightarrow P\square_D Q$ and $G: D \longrightarrow Q\square_C P$ fitting in the following commutative diagrams:

$$\begin{array}{ccc} P & \xrightarrow{\ \sim\ } & P\square_D D \\ {\scriptstyle\sim}\big\downarrow & & \big\downarrow{\scriptstyle 1\square G} \\ C\square_C P & \xrightarrow{F\square 1} & P\square_D Q\square_C P \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{\ \sim\ } & Q\square_C C \\ {\scriptstyle\sim}\big\downarrow & & \big\downarrow{\scriptstyle 1\square F} \\ D\square_D Q & \xrightarrow{G\square 1} & Q\square_C P\square_D Q \end{array}$$

The context is said to be **strict** if both F and G are injective (equivalently, isomorphic). In this case we say that C is M-T equivalent to D, denoted by $C \sim D$.

H-(CO)MODULE COALGEBRAS: *Let H be a Hopf algebra, C a coalgebra. C is said to be a right H-module coalgebra if*

(i) *C is a right H-module,*
(ii) $\Delta(c \leftharpoonup h) = \sum c_{(1)} \leftharpoonup h_{(1)} \otimes c_{(2)} \leftharpoonup h_{(2)}$, $c \in C, h \in H$,
(iii) $\varepsilon(c \leftharpoonup h) = \varepsilon(c)\varepsilon(h)$.

Dually, a coalgebra D is called a left H-comodule coalgebra if

(i) *C is a left H-comodule,*
(ii) $\sum c_{\langle -1\rangle} \otimes \Delta(c_{\langle 0\rangle}) = \sum c_{(1)\langle -1\rangle}c_{(2)\langle -1\rangle} \otimes c_{(1)\langle 0\rangle} \otimes c_{(2)\langle 0\rangle}$,
(iii) $\sum \varepsilon(c_{\langle 0\rangle})c_{\langle -1\rangle} = \varepsilon(c)1_H$.

If H is a finite dimensional Hopf algebra, a coalgebra C is a right H-module coalgebra if and only if C is a left H^*-comodule coalgebra. On the other hand, for any Hopf algebra H and right H-module coalgebra C, the convolution algebra C^* is a left H-module algebra with H-module structure induced by transposition.

Let C be a left H-comodule coalgebra. We may form a smash coproduct coalgebra $C \rtimes H$ cf. [7] which has counit $\varepsilon_C \rtimes \varepsilon_H$ and comultiplication as follows:

$$\Delta(c \rtimes h) = \sum (c_{(1)} \rtimes c_{(2)\langle -1\rangle}h_{(1)}) \otimes (c_{(2)\langle 0\rangle} \rtimes h_{(2)}).$$

If H is finite dimensional, C^* is a left H^*-module algebra. We have the usual smash product algebra $C^*\#H^*$. It is easy to see that $C^*\#H^*$ is exactly the convolution algebra $(C \rtimes H)^*$.

Let E be a coalgebra. A coalgebra F is called a **coextension** of E if E is a quotient coalgebra of F. Let C be a right H-module coalgebra, H a Hopf algebra. Denote by H^+ the augumentation ideal $\operatorname{Ker}\varepsilon$ which is a Hopf ideal. Then $CH^+ = C \leftharpoonup H^+$ is a coideal of C, and C/CH^+ is a coalgebra with a trivial right H-module structure. Let R be the quotient coalgebra C/CH^+. It is not hard to check that R^* is the invariant subalgebra of the left H-module algebra C^*. Dual to the terminology of 'H-extension', we call C/R an H-coextension. View C as a left and right R-comodule. There is a canonical linear map

$$\beta\colon C \otimes H \longrightarrow C\Box_R C, \quad c \otimes h \mapsto \sum c_{(1)}\Box c_{(2)} \leftharpoonup h.$$

If β is bijective, then C/R is said to be an H-Galois coextension cf. [10].

Finally, we recall from [15] the notion of Azumaya coalgebra and the Brauer group of a cocommutative coalgebra. Given two coalgebra maps

$$E \xleftarrow{\mu} C \xrightarrow{\eta} D$$

we say that (D, η) cocommutes with (E, μ) (simply, D cocommutes with E or η cocommutes with μ) if

$$\sum \eta(c_{(1)}) \otimes \mu(c_{(2)}) = \sum \eta(c_{(2)}) \otimes \mu(c_{(1)}), \quad \forall c \in C.$$

Given a coalgebra map η: $C \longrightarrow D$, there exists a cocommutative coalgebra $Z_C(D)$ and a surjective coalgebra map η^d: $C \longrightarrow Z_C(D)$ which cocommutes with η and satisfies the universal mapping property: for any coalgebra map μ: $C \longrightarrow E$ which cocommutes with η, there exists a unique coalgebra map ν: $Z_C(D) \longrightarrow E$ such that $\mu = \nu\eta^d$. $(Z_C(D), \eta^d)$ (simply $Z_C(D)$) is called the **cocommutator coalgebra** of the coalgebra map η. In fact, $Z_C(D) = h_{D^{op}\otimes D}(D, C)$. In particular, the cocommutator coalgebra of 1_D: $D \longrightarrow D$ is a cocommutative quotient coalgebra of D, denoted by $Z(D)$, which is referred to as the **cocenter** of D. A cocommutative quotient coalgebra E of a coalgebra C is said to be **maximal** if $Z_C(E) = E$. Equivalently, if F is a cocommutative quotient coalgebra of C such that $C \longrightarrow F \longrightarrow E$ equals $C \longrightarrow E$, then $F = E$.

Let R be a cocommutative coalgebra. An R-coalgebra D is a k-coalgebra with a coalgebra map ϵ: $D \longrightarrow R$, called the R-counit, which cocommutes with the identity map 1_D. Any coalgebra D is a $Z(D)$-coalgebra. An R-coalgebra D is said to be **cocentral** if $Z(D) \cong R$. An R-coalgebra D is **coseparable** if the comultiplication map Δ: $D \longrightarrow D\Box_R D$ splits as a (D, D)-bicomodule map, or equivalently, D as a left $D^{op}\Box_R D$-comodule with the comodule structure as follow:

(4) $$D \longrightarrow (D^{op}\Box_R D) \otimes D, \; d \mapsto \sum (d_{(3)}\Box d_{(1)}) \otimes d_{(2)},$$

is injective. If D is an R-coseparable coalgebra, then the 'Maschke theorem' cf. [15, 3.4] says that if a D-comodule X is injective as an R-comodule, then X is injective as a D-comodule.

An Azumaya R-coalgebra is defined to be an R-cocentral and R-coseparable coalgebra. An R-coalgebra D is Azumaya if and only if D_R is a quasi-finitely injective cogenerator and $e_R(D) \cong D^e$ canonically [15, 3.14]. If D is an Azumaya R-coalgebra, then the functor $-\square_R D$ defines a Morita–Takeuchi equivalence between the comodule category $\mathbf{M}^R$ and the comodule category $\mathbf{M}^{D^e}$, where $D^e = D^{op}\square_R D$ is said to be the R-enveloping coalgebra of D. Denote by $B(R)$ the set of the isomorphism classes of Azumaya R-coalgebras. Then the Morita–Takeuchi equivalence is an equivalence $\sim$ in $B(R)$. The set $B(R)/\sim$ is a group with multiplication $\square_R$, and called the **Brauer group** of coalgebra R, denoted by $\mathrm{Br}(R)$. It was shown in [15] that two Azumaya coalgebras C and D are equivalent if and only if there are two quasi-finite injective cogenerator R-comodules M, N such that

$$C\square_R\, e_R(M) \cong D\square_R\, e_R(N),$$

where $e_R(M), e_R(N)$ are Azumaya R-coalgebra with the R-counits induced by the bicomodules structures of ${}_RM_R, {}_RN_R$ cf. (3) respectively.

Let $f\colon C \longrightarrow R$ be a coalgebra map of cocommutative coalgebras. The map f induces a homomorphism from Brauer group $\mathrm{Br}(R)$ to $\mathrm{Br}(C)$ [15]:

$$\widetilde{f}\colon \mathrm{Br}(R) \longrightarrow \mathrm{Br}(C), \quad [D] \mapsto [D\square_R C].$$

Denote by $\mathrm{Br}(C/R)$ the kernel of the homomorphism $\widetilde{f}$. $\mathrm{Br}(C/R)$ is called the relative Brauer group (a subgroup of $\mathrm{Br}(R)$) with respect to the map f. The elements in $\mathrm{Br}(C/R)$ are called the split Azumaya R-coalgebras.

2. Hopf Galois coextensions of a cocommutative coalgebra

In this section, H is a finite dimensional Hopf algebra. C is a right H-module coalgebra, or equivalently a left H^*-comodule coalgebra. Let R be the quotient coalgebra C/CH^+. C/R is an H-coextension. C may be viewed as a left or right R-comodule in a natural way. Let $C \rtimes H^*$ be the smash coproduct. There is a left coaction of $C \rtimes H^*$ on C:

$$\chi^l(c) = \sum (c_{(1)} \rtimes c_{(2)\langle -1\rangle}) \otimes c_{(2)\langle 0\rangle}$$

that makes C into a $(C \rtimes H - R)$-bicomodule. Let T be a left integral of H and let λ be the distinguished group-like element of H^* which satisfies:

$$Th =< g, \lambda > T, \quad \forall h \in H.$$

In [3] we defined a right coaction of $C \rtimes H^*$ on C as follows:

$$\chi^r(c) = \sum c_{(1)\langle 0\rangle} \otimes (c_{(2)\langle 0\rangle} \rtimes S^{-1}(c_{(1)\langle -1\rangle} c_{(2)\langle -1\rangle})\lambda)$$

which makes C into a right $C \rtimes H^*$-comodule and an $(R - C \rtimes H)$-bicomodule. Let $\Box$ denote the cotensor product over R. We have a Morita–Takeuchi context:

$$(C \rtimes H^*, R, {}_{C\rtimes H^*}C_R, {}_RC_{C\rtimes H^*}, F, G), \tag{5}$$

where the canonical bicolinear maps are defined as follows:

$$F: C \rtimes H^* \longrightarrow C\Box C, \quad c \rtimes h^* \mapsto \sum c_{(1)} \Box c_{(2)\langle 0\rangle} \langle T, c_{(2)\langle -1\rangle} h^* \rangle,$$
$$G: R \longrightarrow C\Box_{C\rtimes H^*} C, \quad \bar{c} \mapsto \sum c_{(1)\langle 0\rangle} \Box c_{(2)\langle 0\rangle} \langle T, c_{(1)\langle -1\rangle} c_{(2)\langle -1\rangle} \rangle.$$

The relationship between the M-T context (5) and the Galois coextension is that C/R is H-Galois if and only if the canonical map F in (5) is injective cf. [3]. We need the following lemma:

LEMMA 2.1 (cf. [3]): *Let C/R be an H-Galois coextension. Then*
(i) *C_R and ${}_RC$ are quasi-finite injective comodules.*
(ii) *$e_{-R}(C) \cong C \rtimes H^*$ and $e_{R-}(C) \cong C \rtimes H^*$.*

Let D be a coalgebra, M be a right D-comodule. Then there exists a minimal subcoalgebra $D(M)$ of D such that $M = \rho^{-1}(M \otimes D(M)) \simeq M \Box_D D(M)$. Clearly, $D(M) = \bigcap_{i\in I} D_i$, where $M = \rho^{-1}(M \otimes D_i)$. I is not empty since D is some D_i. If we choose a linear basis $\{m_\lambda, \lambda \in \Lambda\}$ of M and $\rho(m_\lambda) = \sum_\mu m_\mu \otimes e_{\lambda\mu}$, $\lambda, \mu \in \Lambda$, then $D(M)$ is generated by the set $\{e_{\lambda\mu}, \lambda, \mu \in \Lambda\}$. M is said to be (D-)**cofaithful** if $D = D(M)$.

In the sequel, we discuss an H-Galois coextension C/R with R being cocommutative; this will be used in the next section.

LEMMA 2.2: *Let R be a cocommutative coalgebra, M an R-comodule. If R' is a subcoalgebra of R such that $M\Box R' = 0$, then $R' \bigcap R(M) = 0$.*

Proof: Suppose that $R' \bigcap R(M) \neq 0$, hence there exists a simple subcoalgebra $S \subseteq R' \bigcap R(M)$. Obviously, $M \Box S = 0$. Since $R(M)$ is a direct sum $\bigoplus_i E_i$ of irreducible subcoalgebras E_i, there exists some E_j which contains S. Then $M\Box E_j$ must be zero. Otherwise, we would have that $M(E_j) = \rho^{-1}(M \otimes E_j) \simeq M\Box E_j$ is a non-zero E_j-comodule. There is a simple E_j-subcomodule $V \subsetneq M(E_j)$. However, by [5, p. 354] there exists a simple subcoalgebra S' of E_j such that $V \longrightarrow V \otimes S'$ is the restriction of $M(E_j) \longrightarrow M(E_j) \otimes E_j$. But since E_j is irreducible, $S' = S$. Therefore, we have

$$V = V\Box_S S = V\Box_{E_j} S \subseteq M(E_j)\Box_{E_j} S = M\Box S = 0,$$

leading to a contradiction! This yields:

$$M = M(R(M)) = \bigoplus_i M(E_i) = \bigoplus_{i\neq j} M(E_i) = M(E'),$$

where $E = \bigoplus_{i\neq j} E_i$. This contradicts the choice of $R(M)$. Therefore we must have $R' \bigcap R(M) = 0$. ∎

Corollary 2.3: *Let R be a cocommutative coalgebra. If P is a quasi-finite injective R-comodule, then P_R is a cogenerator if and only if P_R is cofaithful.*

Proof: Since P_R is quasi-finite injective we have the canonical isomorphism cf. [14, 2.5,3.2]:

$$F\colon e_R(P) \longrightarrow P\Box P_*,\ P_* = h_R(P, R).$$

By [14, 3.2], P_R is a cogenerator if and only if the canonical map:

$$G\colon R \longrightarrow P_*\Box_{e_R(P)} P$$

is injective. Suppose that P_R is a cogenerator. We may cotensor the isomorphism G by $R(P)$ over R, then we get an R-colinear isomorphism:

$$G\Box R(P)\colon R(P) \simeq R\Box R(P) \simeq P_*\Box_{e_R(P)} P\Box R(P) \simeq P_*\Box_{e_R(P)} P,$$

where $P\Box R(P) \simeq P$ by definition. It follows that the inclusion $R(P) \longrightarrow R$ is an isomorphism, and hence $R(P) = R$.

Conversely, let K be the kernel of G. Then we have the following exact sequence:

$$0 \longrightarrow P\Box K \longrightarrow P\Box R \stackrel{1\Box G}{\longrightarrow} P\Box P_*\Box_{e_R(P)} P.$$

Moreover, the following diagram is commutative:

$$\begin{array}{ccc} P & \xrightarrow{\simeq} & e_R(P)\square_{e_R(P)}P \\ \simeq\downarrow & & \downarrow F\square 1 \\ P\square R & \xrightarrow{1\square G} & P\square P_*\square_{e_R(P)}P \end{array}$$

Since $F\square 1$ is an isomorphism, $1\square G$ is an isomorphism too, and $P\square K = 0$. By Lemma 2.2, $K\bigcap R(P) = 0$. However, $R(P) = R$, hence $K = 0$, and G is injective. ∎

In the sequel, a quasi-finite injective cogenerator comodule over a cocommutative coalgebra is said to be a **cofaithfully injective** comodule.

COROLLARY 2.4: *Let C/R be an H-Galois coextension. If R is cocommutative then $\mathbf{M}^R \sim \mathbf{M}^{C\rtimes H}$.*

Proof: By assumption, C_R is a quasi-finite injective comodule. Now R is cocommutative and C_R is cofaithful. It follows from Proposition 2.3 that C_R is a cogenerator, and hence the functor $-\square C$ defines a M-T equivalence from $\mathbf{M}^R$ to $\mathbf{M}^{C\rtimes H}$. ∎

PROPOSITION 2.5: *If C/R is an H-coextension and R is cocommutative, then C/R is an H-Galois coextension if and only if C^*/R^* is H^*-Galois. In this case, C is a finitely cogenerated and free R-comodule.*

Proof: The 'if part' holds in the general case cf. [3]. We only need to show that C^*/R^* is H^*-Galois if C/R is H-Galois, and we prove the equivalent condition that C^* as a left $C^*\#H$-module is a generator. If C as an R-comodule is finitely cogenerated we have an $C\rtimes H^*$-colinear composite map:

$$C\rtimes H^* \simeq C\square C \hookrightarrow C\square R^n \simeq C^n.$$

Dualizing the above map, we obtain the desired $C^*\#H$-linear surjective map $C^{*n} \longrightarrow C^*\#H \longrightarrow 0$, and C^* is a $C^*\#H$-generator.

We show that C_R is finitely cogenerated. Let $R = \bigoplus_i R_i$, a direct sum of irreducible subcoalgebras. Identifying C with $C\square R$, we may decompose C as follows:

$$C = C\square(\bigoplus_i R_i) = \bigoplus_i C\square R_i = \bigoplus C_i, C_i = C\square R_i.$$

It is clear that C_i is an R_i-coalgebra, and a right H-module coalgebra that satisfies:

$$C_iH^+ = C_i \leftharpoonup H^+ = C \leftharpoonup H^+\Box R_i.$$

Since R_i is a direct summond of R, R_i is an injective (or equivalently coflat) R-comodule. It follows that $-\Box R_i$ is an exact functor. So we have an exact sequence:

$$0 \longrightarrow CH^+\Box R_i \longrightarrow C\Box R_i \longrightarrow R\Box R_i \longrightarrow 0.$$

This is exactly the exact sequence:

$$0 \longrightarrow C_iH^+ \longrightarrow C_i \longrightarrow R_i \longrightarrow 0.$$

Thus, for each i, C_i/R_i is an H-coextension. We show that C_i/R_i is H-Galois. Indeed, the canonical Galois isomorphism β may be cotensored by R_i, so as to yield an isomorphism:

$$\beta\Box R_i\colon C\otimes H\Box R_i \longrightarrow C\Box C\Box R_i,$$

which is exactly the following isomorphism:

$$\beta_i\colon (C\Box R_i)\otimes H \longrightarrow (C\Box R_i)\Box_{R_i}(C\Box R_i).$$

So C_i/R_i is H-Galois. Now R_i is irreducible. By Lemma 2.1 C_i is a (quasi-finite) injective R_i-comodule, and hence a free R_i-comodule. This implies that C_i is finitely cogenerated (because a quasi-finite free comodule must be finitely cogenerated). Assume $C_i \simeq R_i \otimes W$, $\dim(W) = n$ for some positive integral n. Then the Galois isomorphism β_i induces an isomorphism

$$R_i\otimes H\otimes W \simeq R_i\otimes W\otimes W. \tag{6}$$

Note that R_i is irreducible and the dimension of the coradical of R_i is finite. Since R_i is cocommutative the coradical of the regular R_i-comodule is the same as the coradical of the coalgebra R_i. Computing the dimensions of the coradicals in the two sides of (6) we arrive at $n = \dim W = \dim H$. As n depends only on H, we have that

$$C = \bigoplus_i C_i \simeq \bigoplus_i R_i^n \simeq (\bigoplus R_i)^n = R^n$$

is a free R-comodule and it is clearly finitely cogenerated. ∎

LEMMA 2.6: *Let C/R be an H-Galois coextension. If C is cocommutative and H is cosemisimple, then C is an R-coseparable coalgebra.*

Proof: Let $\beta\colon C \otimes H \longrightarrow C \square C$ be the canonical Galois isomorphism. Since C is cocommutative, β is a coalgebra isomorphism. Define a left $C \otimes H$-comodule structure on C as follows:

$$\chi(c) = \sum (c_{(1)} \otimes 1) \otimes c_{(2)}.$$

One easily sees that the comodule structure χ induces the usual left C^e-comodule structure ρ of C by means of the base change β, that is , the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\chi} & (C \otimes H) \otimes C \\ & \searrow^{\rho} & \downarrow{\scriptstyle \beta\otimes 1} \\ & & (C \square C) \otimes C \end{array}$$

Now the usual embedding $C \longrightarrow C \otimes H$, $c \mapsto c \otimes 1$ is left $C \otimes H$-colinear. Since H is cosemisimple, we may choose an integral $T \in H^*$ such that $\langle T, 1_H \rangle = 1_k$. Define a map

$$\kappa\colon C \otimes H \longrightarrow C, \quad c \otimes h \mapsto c\langle T, h \rangle.$$

Note that for any $h \in H$, $\langle T, h \rangle 1_H = \sum h_{(1)} \langle T, h_{(2)} \rangle$, so κ is a left $C \otimes H$-comodule map. Of course κ is surjective. Thus, C as a left $C \otimes H$-comodule is a direct summand of the regular $C \otimes H$-comodule $C \otimes H$, and hence is injective. By a base change argument it follows that C is an injective C^e-comodule, i.e. C is an R-coseparable coalgebra. ∎

Schneider recently proved in a different way that the non-cocommutative Galois coextension is coseparable if the Hopf algebra is cosemisimple [11].

3. Galois cohomology

In this section, we will show that the relative Brauer group of a Galois coextension is the second Galois cohomological group, that is, we prove an analogue of the crossed product theorem. First let us recall the Sweedler's cohomology and the Doi cohomology. Let H be a commutative Hopf algebra, C a cocommutative left H-comodule coalgebra. The Doi cohomology of H in C is defined by means of the semi-cosimplicial complex cf. [4, 6]:

$$\mathrm{Reg}(C, H) \xrightarrow{d_1} \mathrm{Reg}(C, H^2) \xrightarrow{d_2} \cdots \longrightarrow \mathrm{Reg}(C, H^n) \xrightarrow{d_n} \mathrm{Reg}(C, H^{n+1}) \longrightarrow \cdots$$

where $\mathrm{Reg}(C, H^n)$ is the unit group of the convolution algebra $\mathrm{Hom}(C, H^n)$ cf. [13], and the n-th degree differential d_n: $\mathrm{Reg}(C, H^n) \longrightarrow \mathrm{Reg}(C, H^{n+1})$ is defined as $\overline{f}_0 * \overline{f}_1^{-1} * \cdots * \overline{f}_{n-1}^{\pm 1}$, and the coface map $\overline{f}_i$ is induced by the coface map

$$f_i\colon H^n \longrightarrow H^{n+1}, \quad f_i(h_0 \otimes h_1 \otimes \cdots \otimes h_{n-1}) = h_0 \otimes \cdots \otimes \Delta h_i \otimes \cdots \otimes h_{n-1},$$

and $f_n(h_0 \otimes \cdots \otimes h_{n-1}) = h_0 \otimes \cdots \otimes h_{n-1} \otimes 1$. The n-th cohomology group of H in C is defined by

$$H^n(C, H) = \mathrm{Ker}\, d_n / \mathrm{Im}\, d_{n-1} = Z^n(C, H)/B^n(C, H).$$

If H is a cocommutative Hopf algebra and A is a commutative left H-module algebra, the Sweedler cohomology of H in A is defined from the semi-cosimplicial complex cf. [12]:

$$\mathrm{Reg}(H, A) \xrightarrow{d^1} \mathrm{Reg}(H^2, A) \xrightarrow{d^2} \cdots \longrightarrow \mathrm{Reg}(H^n, A) \xrightarrow{d^n} \mathrm{Reg}(H^{n+1}, A) \longrightarrow \cdots$$

where $d^n = \partial^1 * \partial^{2^{-1}} * \cdots * \partial^{n \pm 1}$, and the coface map

$$\partial^i\colon \mathrm{Reg}(H^n, A) \longrightarrow \mathrm{Reg}(H^{n+1}, A)$$

is induced by the face map

$$\partial_n\colon H^{n+1} \longrightarrow H^n, \quad h_0 \otimes \cdots \otimes h_n \mapsto h_0 \otimes \cdots \otimes h_i h_{i+1} \otimes \cdots h_n,$$

where $\partial_n(h_0 \otimes \cdots \otimes h_n) = h_0 \otimes \cdots \otimes h_{n-1}\varepsilon(h_n)$. The Sweedler cohomology group is defined by

$$H^n(H, A) = \mathrm{Ker}\, d^n / \mathrm{Im}\, d^{n-1}.$$

Let H be a finite dimensional commutative Hopf algebra, C a cocommutative left H-comodule coalgebra. Then C^* is a left H^*-module algebra. Of course C^* is commutative and H^* is cocommutative. As explained above, we may consider the Doi cohomology group $H^n(C, H)$ and the Sweedler cohomology group $H^n(H^*, C^*)$. We have

LEMMA 3.1: *Suppose that H is a finite dimensional commutative Hopf algebra, and C is a cocommutative left H-comodule coalgebra. Then $H^n(C, H) \cong H^n(H^*, C^*)$, $\forall n$.*

Proof: Define morphisms ϕ_n as follows:

$$\mathrm{Reg}(C, H^n) \longrightarrow \mathrm{Reg}(H^{*n}, C^*), \quad \langle \phi_n(f)(x^{\otimes n}), c\rangle = \langle x^{\otimes n}, f(c)\rangle,$$

where $\langle , \rangle\colon V^* \otimes V \longrightarrow k$ is the usual bilinear map for a vector space V, and $x^{\otimes n} \in H^{*n}$. One may easily show that ϕ is a group morphism. In fact, ϕ is an isomorphism with inverse map ψ given as follows:

$$\langle \psi(g)(c), x^{\otimes n}\rangle = \langle g(x^{\otimes n}), c\rangle, \quad c \in C, \quad x^{\otimes n} \in H^{*n};$$

here we identify H with H^{**}. Now it is straightforward to check that the following diagram commutes:

$$\begin{array}{ccccccccc}
\mathrm{Reg}(C,H) & \xrightarrow{d_1} & \mathrm{Reg}(C,H^2) & \xrightarrow{d_2} & \cdots & \longrightarrow & \mathrm{Reg}(C,H^n) & \xrightarrow{d_n} & \cdots \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow & & \downarrow{\scriptstyle \phi_n} & & \\
\mathrm{Reg}(H^*,C^*) & \xrightarrow{d^1} & \mathrm{Reg}(H^{*2},C^*) & \xrightarrow{d^2} & \cdots & \longrightarrow & \mathrm{Reg}(H^{*n},C^*) & \xrightarrow{d^n} & \cdots
\end{array}$$

This yields that $\mathrm{Ker}\, d_n \simeq \mathrm{Ker}\, d^n$, $\mathrm{Im}\, d_{n-1} \simeq \mathrm{Im}\, d^n$ and $H^n(C,H) \simeq H^n(H^*,C^*)$. Note that via the above isomorphisms normalized cocycles correspond to normalized ones. ∎

Now we focus on Galois cohomology. In the sequel, G **always denotes a finite group. If no confusion arises, a** kG**-Galois coextension** C/R **means that** C/R **is a cocommutative Galois coextension**. Let C be a cocommutative right kG-module coalgebra (G acts on C by automorphisms). This is equivalent to say that C is a cocommutative left H-comodule coalgebra, where $H = (kG)^*$. So C^* is a left kG-module algebra. It is well-known that the Galois cohomology coincides with the Sweedler cohomology when the Hopf algebra is a group Hopf algebra. By Lemma 3.1, we have that the Doi cohomology group $H^n(C,H)$ is isomorphic to the Galois cohomology group $H^n(G,C^*)$.

Let C_i/R be a kG_i-Galois coextension, where the G_i are finite groups, $i = 1,2$. Then $C_1\square C_2/R$ is a $k(G_1 \times G_2)$-Galois coextension because of the following commutative diagram:

$$\begin{array}{ccc}
(C_1\square C_2)\otimes k(G_1\times G_2) & \xrightarrow{\beta} & (C_1\square C_2)\square(C_1\square C_2) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
(C_1\otimes kG_1)\square(C_2\otimes kG_2) & \xrightarrow{\beta_1\square\beta_2} & (C_1\square C_1)\square(C_2\square C_2)
\end{array}$$

Since the R-comodules C_i are finitely cogenerated and free (cf. Prop. 2.5) we have that $(C_1\square C_2)^* \cong C_1^* \otimes_{R^*} C_2^*$. This gives rise to an isomorphism between the two cohomology groups:

$$H^n(C_1\square C_2, k(G_1\times G_2)^*) \simeq H^n(G_1\times G_2, C_1^*\otimes_{R^*} C_2^*).$$

It is clear that if $f_1 \in Z^n(C, kG_1^*), f_2 \in Z^n(C_2, kG_2^*)$ then

$$f_1 \Box f_2 \in Z^n(C_1 \Box C_2, k(G_1 \times G_2)^*) \quad \text{and} \quad \overline{f}_1 \Box \overline{f}_2 \in H^n(C_1 \Box C_2, k(G_1 \times G_2)^*).$$

On the other hand, if C/R is a kG-Galois coextension, then $C \Box C/C$ is a $k(G \times 1)$-Galois coextension and a $k(1 \times G)$-Galois coextension. This can be seen from the commutative diagram below:

$$\begin{array}{ccc} (C \Box C) \otimes k(G \times 1) & \xrightarrow{\beta_{C \Box C}} & (C \Box C) \Box_C (C \Box C) \\ \simeq \downarrow & & \downarrow \simeq \\ (C \otimes kG) \Box (C \otimes k) \xrightarrow{\beta_C \Box 1} & (C \Box C) \Box C & \longrightarrow C \Box (C \Box C). \end{array} \tag{7}$$

Now we look at the second Doi cohomology group.

Lemma 3.2: *Let C/R be a kG-Galois coextension and $f \in Z^2(C, kG^*)$ a 2-cocycle. Then $f \Box 1 \in Z^2(C \Box C, k(G \times 1)^*)$, $1 \Box f \in Z^2(C \Box C, k(1 \times G)^*)$ and $f \Box f^{-1} \in Z^2(C \Box C, k(G \times G)^*)$. If $f, g \in Z^2(C, kG^*)$ then $f \Box g$ is cohomologous to $fg \Box 1$ in $Z^2(C \Box C, k(G \times G)^*)$.*

Proof: By Proposition 2.5 C^*/R^* is an kG^*-Galois extension, and C as an R-comodule is finitely cogenerated and free. So the natural map $(C \Box C)^* \longrightarrow C^* \otimes_{R^*} C^*$ is an isomorphism. In Lemma 3.1 the isomorphism ϕ_2 restricts to the following two group isomorphisms:

$$Z^2(C \Box C, H \otimes k) \cong Z^2(G \times 1, C^* \otimes_{R^*} C^*),$$
$$Z^2(C \Box C, H \otimes H) \cong Z^2(G \times G, C^* \otimes_{R^*} C^*),$$

where $H = kG^*$. One easily checks that if f, g correspond to $\overline{f}, \overline{g}$ respectively under the isomorphism ϕ_2 in Lemma 3.1, then

$f \Box 1$ corresponds to $\overline{f} \otimes 1$,

$1 \Box f$ corresponds to $1 \otimes \overline{f}$,

$f \Box g$ corresponds to $\overline{f} \otimes \overline{g}$, and

$fg \Box 1$ corresponds to $\overline{f}\overline{g} \otimes 1$ respectively.

Now applying [2, A.11], we obtain the statements of the lemma. ∎

Recall from [4, 6] that each normalized 2-cocycle f determines a crossed coproduct $C \rtimes_f kG^*$, denoted by $\nabla(f, C, G)$, which has comultiplication given as follows: $\forall c \in C, h \in kG^*$,

$$\Delta(c \rtimes h) = \sum (c_{(1)} \rtimes c_{(2)\langle -1 \rangle} f_1(c_{(3)}) h_{(1)}) \otimes (c_{(2)\langle 0 \rangle} \rtimes f_2(c_{(3)}) h_{(2)}),$$

and counit $\varepsilon_C \rtimes \varepsilon_{kG^*}$, where $f(c) = \sum f_1(c) \otimes f_2(c)$. It is well-known that $\nabla(f, C, G)$ is isomorphic to $\nabla(g, C, G)$ if and only if f is cohomologous to g. In the sequel, if no confusion can arise, we let ∇ denote the crossed coproduct $\nabla(f, C, G)$. Observe that we have a C^*-linear algebra isomorphism:

$$\nabla(f, C, G)^* \cong \Delta(G, C^*, \overline{f}) = C^* \otimes_{\overline{f}} G, \quad \overline{f} = \phi_2(f). \tag{8}$$

PROPOSITION 3.3: *Let C/R be a kG-Galois coextension. If $f \in Z^2(C, kG^*)$ is a normalized cocycle, then $\nabla(f, C, G)$ is an Azumaya R-coalgebra and $Z_\nabla(C) = C$ with respect to the canonical map $\nabla \longrightarrow C, c \rtimes p \mapsto p(1)c, c \in C, p \in kG^*$, that is, C is a maximal cocommutative quotient coalgebra of ∇.*

Proof: It is clear that $\nabla(f, C, G)$ is an R-coalgebra. The definition of the crossed coproduct entails the existence of a C-coalgebra isomorphism:

$$\nabla(f, C, G)\square C \cong \nabla(f\square 1, C\square C, G \times 1).$$

Now $f\square 1$ is cohomologous to 1 by Lemma 3.2. It follows that

$$\nabla(f\square 1, C\square C, G \times 1) \cong \nabla(1, C\square C, G \times 1) = (C\square C) \rtimes k(G \times 1)^*.$$

Since $C\square C/C$ is $k(G \times 1)$-Galois (cf. (7)), it follows from Lemma 2.1 that we have C-coalgebra isomorphisms:

$$\nabla(f, C, G)\square C \cong (C\square C) \rtimes k(G \times 1)^* \cong e_C(C\square C). \tag{9}$$

Now ${}_C(C\square C)$ is quasi-finite injective by Lemma 2.1 and a cogenerator because it is free (cf. Prop. 2.5). $e_C(C\square C)$ is a C-Azumaya coalgebra, i.e. a C-coseparable cocentral C-coalgebra. Thus $\nabla(f, C, G)\square C$ is C-coseparable and C-cocentral by (9). Since C is R-coseparable cf. Lemma 2.6, by [15, Prop. 3.6], $\nabla(f, C, G)\square C$ is R-coseparable. Now R is a direct summand of C as an R-comodule since C is R-free (cf. Prop. 2.5). It follows from [15, Prop. 3.3] that $\nabla(f, C, G)$ is R-coseparable. Let Z be the cocenter of $\nabla(f, C, G)$ which is clearly an R-coalgebra. Then $Z\square C$ is the cocenter of $\nabla(f, C, G)\square C$ by [15, 3.10]. But the cocenter of $\nabla(f, C, G)\square C$ is C. This implies that the map $-\square C$: $Z\square C \longrightarrow R\square C$ is an isomorphism. By Corollary 2.4, ${}_R C$ is faithfully coflat and hence the counit $Z \longrightarrow R$ is an isomorphism, that is, R is the cocenter of $\nabla(f, C, G)$.

Let D be the cocommutator coalgebra $Z_\nabla(C)$. Since C cocommutes with C there is a unique coalgebra map $D \longrightarrow C$ such that the following diagram commutes:

$$\begin{array}{ccc} & \nabla(f,C,G) & \\ \swarrow & & \searrow \\ D & \longrightarrow & C \end{array}$$

Dualizing the above diagram we arrive at an injective algebra morphism:

$$C^* \hookrightarrow D^* \hookrightarrow \nabla^*,$$

where C^* commutes with D^*. However, C^* is a maximal commutative subalgebra of the Azumaya algebra ∇^* by (8) and [2, Th. A.12]. Hence $C^* = D^*$, and $C = D$. ∎

Let C be a cocommutative R-coalgebra. If P is a C-comodule such that the restricted R-comodule P_R is quasi-finite, then the bicomodule ${}_CP_R$ induces an R-coalgebra map ϵ: $e_R(P) \longrightarrow C$. If P_C is quasi-finite, then ϵ has a nice cocommutator coalgebra.

LEMMA 3.4: *Let C be a cocommutative R-coalgebra, P a quasi-finite C-comodule such that P_R is also quasi-finite. Let ϵ: $e_R(P) \longrightarrow C$ be as above. Then $Z_{e_R(P)}(C) = e_C(P)$.*

Proof: The $(e_C(P) - C)$-bicomodule structure of P induces an $(e_C(P) - R)$-bicomodule structure on P which induces a canonical coalgebra map π: $e_R(P) \longrightarrow e_C(P)$. Given any quotient coalgebra map μ: $e_R(P) \longrightarrow E$ such that μ cocommutes with ϵ. This gives rise to an $(E - C)$-bicomodule structure on P as follows:

$$\rho_E(p) = (\mu \otimes 1)\rho(p), \quad \rho_C(p) = t(\epsilon \otimes 1)\rho(p), \quad p \in P$$

where ρ is the comodule structure of ${}_{e_R(P)}P$, and t is the switch map. It makes sense because μ cocommutes with ϵ. It follows that there exists a coalgebra map ν: $e_C(P) \longrightarrow E$ such that $\mu = \nu\pi$. This means that $e_C(P)$ satisfies the universal property of $Z_{e_R(P)}(C)$, so that $(e_C(P), \pi)$ is the desired cocommutator coalgebra. ∎

Let D be an R-coalgebra. D may be regarded as a left D^e-comodule in a natural way cf. (4). If C is a cocommutative quotient coalgebra of D, then the above left D^e-comodule structure restricts to a left $C \square D$-comodule structure on

D. Since C is cocommutative, D may be adorned with a $(C\Box D-C)$-bicomodule structure:

$$D \longrightarrow (C\Box D)\otimes D\otimes C,\ d\mapsto \sum(\overline{d_{(3)}}\Box d_{(1)})\otimes d_{(2)}\otimes \overline{d_{(4)}}, \tag{10}$$

where $\overline{x}$ denotes the the image in C of the element $x\in D$.

LEMMA 3.5: *Let D be an Azumaya R-coalgebra, C a maximal cocommutative quotient coalgebra of D. View D as a $(C\Box D-C)$-bicomodule as above. Then the canonical map $j\colon e_C(D)\longrightarrow C\Box D$ induced by the bicomodule structure is a C-colinear isomorphism. If C is R-coseparable then D is a cofaithfully injective C-comodule and an injective left $C\Box D$-comodule.*

Proof: Recall from [15] that for an Azumaya coalgebra D, the induced R-coalgebra map:

$$f\colon e_R(D)\longrightarrow D^{op}\Box D=D^e,$$

given by the bicomodule structure of ${}_{D^e}D_R$, is an isomorphism. Observe that f is a C-colinear map, i.e. the following diagram commutes:

$$\begin{array}{ccc} e_R(D) & \xrightarrow{\ f\ } & D^{op}\Box D \\ & {\scriptstyle\epsilon}\searrow \quad \swarrow{\scriptstyle p\Box\varepsilon} & \\ & C & \end{array}$$

Here p is the projection map. Since C is a maximal quotient cocommutative coalgebra of D as well as of D^{op}, we have $Z_{D^{op}}(C)=C$ and $Z_{D^e}(C)=C\Box D$ (with respect to $p\Box\varepsilon$). On the other hand, D_R is quasi-finite since D is R-Azumaya. By the remark before the *basic lemma*, we have that D_C is quasi-finite. Thus, $e_C(D)$ makes sense, and $Z_{e_R(D)}(C)=e_C(D)$ by Lemma 3.4. So f induces a coalgebra isomorphism:

$$j\colon e_C(D)\longrightarrow C\Box D$$

which is exactly induced by the bicomodule structure of ${}_{C\Box D}D_C$ cf. (3). The R-injectivity of D can be lifted to the C-injectivity because C is R-coseparable cf. (2.6). It is obvious that D_C is cofaithful. ∎

Let C/R be a cocommutative coalgebra coextension. An R-coalgebra A is said to be split by C if there is a cofaithfully injective C-comodule N such that $A\Box C\cong e_C(N)$ as C-coalgebras.

PROPOSITION 3.6: *Let D be an Azuamaya R-coalgebra which is split by a cocommutative coalgebra C. If C/R is a coextension (not necessarily Galois) such that C_R is quasi-finite injective, then there exists an Azumaya R-coalgebra T which is equivalent to D and has C as a maximal cocommutative quotient coalgebra of T.*

Proof: C splits D^{op} because it splits D. Hence there is a cofaithfully injective C-comodule P such that ℓ: $C\Box D^{op} \cong e_C(P)$, where ℓ is a coalgebra map. By Lemma 3.4, we have a surjective coalgebra map:

$$\eta\colon e_R(P) \xrightarrow{\pi} e_C(P) \xrightarrow{\ell^{-1}} C\Box D^{op} \xrightarrow{\varepsilon\Box 1} D^{op}.$$

Let (T, η^d) be the cocommutator coalgebra of the map η. Then the map

$$(\eta^d\Box\eta)\Delta\colon e_R(P) \longrightarrow T\Box D^{op}$$

is a coalgebra isomorphism. It follows from [15, 3.21] that T is an Azumaya R-coalgebra equivalent to D. Let ϵ be the map $e_R(P) \longrightarrow C$ as in Lemma 3.4. Note that ϵ is the composite map

$$e_R(P) \xrightarrow{\pi} e_C(P) \xrightarrow{\iota} C,$$

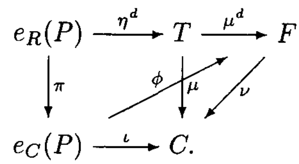
where ι is the canonical map of $e_C(P)$ induced by bicomodule structure of ${}_CP_C$ cf.(3). Now ℓ is a coalgebra map implies that C cocommutes with D^{op}, and hence ϵ cocommutes with η. Thus there is a coalgebra map μ: $T \longrightarrow C$ such that $\epsilon = \mu\eta^d$ and the following diagram commutes:

$$\begin{array}{ccc} e_R(P) & \xrightarrow{\eta^d} & T \\ \Big\downarrow{\scriptstyle\pi} & \searrow^{\epsilon} & \Big\downarrow{\scriptstyle\mu} \\ e_C(P) & \xrightarrow{\iota} & C \end{array}$$

Let (F, μ^d) be the cocommutator coalgebra of coalgebra map μ. Since (C, μ) cocommutes with itself, there is a coalgebra map ν: $F \longrightarrow C$ such that $\mu = \nu\mu^d$. Observe that it follows from the above diagram that the composite coalgebra map

$$e_R(P) \xrightarrow{\eta^d} T \xrightarrow{\mu^d} F$$

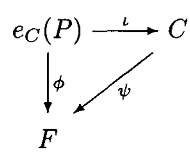
cocommutes with coalgebra map ϵ: $e_R(P) \longrightarrow C$. Since $e_C(P)$ is the cocommutator of ϵ, there exists a unique (surjective) coalgebra map ϕ: $e_C(P) \longrightarrow F$ such that the following diagram commutes:



We claim that ϕ cocommutes with $e_C(P) \xrightarrow{\iota} C$. Indeed, for $y \in e_C(P)$, $y = \pi(x)$ for some element $x \in e_R(P)$, and

$$\begin{aligned}\sum \iota(y_{(1)}) \otimes \phi(y_{(2)}) &= \sum \iota\pi(x_{(1)}) \otimes \phi\pi(x_{(2)}) \\ &= \sum \mu\eta^d(x_{(1)}) \otimes \mu^d\eta^d(x_{(2)}) \\ &= \sum \mu\eta^d(x_{(2)}) \otimes \mu^d\eta^d(x_{(1)}) \\ &= \sum \iota(y_{(2)}) \otimes \phi(y_{(1)}).\end{aligned}$$

Since $e_C(P)$ is C-Azumaya, C is the cocenter of $e_C(P)$. Hence there is a unique (surjective) coalgebra map ψ: $C \longrightarrow F$ such that the following diagram commutes:



It follows from the universal property of the cocenter C that $\nu\psi = 1$. Thus ψ is injective, and hence an isomorphism. ∎

Recall from [15] that a coalgebra automorphism f of a coalgebra D is said to be inner if the transposed algebra automorphism f^* of D^* is inner, that is, there is a unit $p \in D^*$ such that

$$f(d) = \sum p(d_{(1)})d_{(2)}p^{-1}(d_{(3)}), \quad \forall d \in D.$$

An R-comodule N is **invertible** if there exists an R-comodule M such that $N\square_R M \simeq R$. Since R is cocommutative, any invertible comodule over R is isomorphic to R cf. [15, Cor. 3.23].

Proposition 3.7: *Let C/R be a kG-Galois coextension. If D is an Azumaya R-coalgebra and C is a maximal cocommutative quotient coalgebra of D that splits D, then each $\sigma \in G$ can be extended to an inner automorphism of D.*

Proof: Considering D as a C-comodule in the usual way, we have a natural $(C\Box D - C)$-bicomodule structure on D cf. (10). There is however another bicomodule structure on D. In order to distinguish between the two we write $\underline{D}$ instead of D whenever we refer to the second $(C\Box D - C)$-bicomodule structure given by:

$$\chi^l(\underline{d}) = \sum(\sigma(\overline{d_{(2)}})\Box d_{(1)}) \otimes \underline{d_{(2)}} \quad \text{and} \quad \chi^r(\underline{d}) = \sum \underline{d_{(1)}} \otimes \sigma(\overline{d_{(2)}}),$$

where we write $\overline{d}_{(2)}$ for the images of the elements $d_{(2)}$ in C since C is a quotient of D (we will use this symbol later). By Lemma 3.5, the induced coalgebra map $j\colon e_C(D) \longrightarrow C\Box D$ is an isomorphism, and D as a C-comodule is cofaithfully injective. Now take $C' = C$ as coalgebras and let $\sigma\colon C \longrightarrow C'$ be a coalgebra isomorphism. Via base change, D may be viewed as an $(e_C(D) - C')$-bicomodule. So there is a unique coalgebra map $\sigma'\colon e_{C'}(D) \longrightarrow e_C(D)$ such that the comodule structure of ${}_{e_C(D)}D$ comes from ${}_{e_{C'}(D)}D$. It is clear that σ' is an isomorphism which has inverse $\sigma^{-1'}$. Now it is straightforward to check that the following diagram commutes:

$$\begin{array}{ccc} e_C(D) & \xrightarrow{\ j\ } & C\Box D \\ {\scriptstyle \sigma'}\downarrow & & \downarrow{\scriptstyle \sigma\Box 1} \\ e_{C'}(D) & \xrightarrow{\ j'\ } & C'\Box D, \end{array}$$

where j' is the coalgebra map induced by the bicomodule structure of ${}_{C'\Box D}D_{C'}$. By Lemma 3.5, j is an isomorphism, and hence j' is an isomorphism. Moreover, by a base change argument, $\underline{D}$ is a cofaithfully injective C-comodule. Thus, $\underline{D}\Box_C-$ and $D\Box_C-$ are two equivalence functors from the category ${}^C\mathbf{M}$ to the category ${}^{C\Box D}\mathbf{M}$. So there are two C-comodules M, N such that

$$\underline{D} \simeq D\Box_C M \quad \text{and} \quad D \simeq \underline{D}\Box_C N.$$

This implies that $M\Box_C N \simeq C$, that is, M, N are invertible C-comodules. It follows from [15, 3.23] we have $M \simeq C$, and $\underline{D} \simeq D$ as left $C\Box D$-comodules. Let ϕ be the isomorphism from $\underline{D}$ to D. ϕ is, in particular, a left D-comodule isomorphism of D because $\underline{D} = D$ as left D-comodules. Moreover, $\mathrm{Com}_{D-}(D, D) \simeq D^*$ and $\mathrm{Aut}_{D-}(D) \simeq U(D^*)$, the unit group of the convolution algebra D^*. There is an invertible element $p \in D^*$ such that

$$\phi(x) = \sum x_{(1)}p(x_{(2)}) \quad \text{and} \quad \phi^{-1}(x) = \sum x_{(1)}p^{-1}(x_{(2)}), \quad x \in D.$$

On the other hand, ϕ is a C-colinear map. This yields that for any $x \in D$ we have

$$\begin{aligned}\sum \sigma(\overline{x_{(2)}}) \otimes \phi(x_{(1)}) &= \sum \phi(x)_{\langle -1\rangle} \otimes \phi(x)_{\langle 0\rangle} \\ &= \sum \overline{x_{(2)}} \otimes x_{(1)}p(x_{(3)}).\end{aligned}$$

This gives rise to the formula:

$$\sum \sigma(\overline{x_{(2)}})p(x_{(1)}) = \sum \overline{x_{(1)}}p(x_{(2)}).$$

Now for $x \in D$, we have

$$\begin{aligned}\sigma(\overline{x}) &= \sum \sigma(\overline{x_{(3)}})p(x_{(2)})p^{-1}(x_{(1)}) \\ &= \sum p^{-1}(x_{(1)})\overline{x_{(2)}}p(x_{(3)}),\end{aligned}$$

that is, σ is a D-inner isomorphism of C. Therefore σ can be extended to an inner automorphism of D. ∎

Now we are able to show the main result:

THEOREM 3.8: *Let G be a finite group, R a cocommutative coalgebra. If C is cocommutative and a right kG-module coalgebra such that C/R is a kG-Galois coextension, then $H^2(C, kG^*) \cong \mathrm{Br}(C/R)$.*

Proof: The map Φ: $H^2(C, kG^*) \longrightarrow \mathrm{Br}(C/R)$ is defined by $\Phi(\alpha) = \nabla(\alpha, C, G)$. This is well-defined since $\nabla(\alpha, C, G)$ is split by C cf.(9). Suppose that $\nabla(\alpha, C, G) = 1$ in $\mathrm{Br}(R)$. Then there is a cofaithfully injective R-comodule such that as R-coalgebras

$$e_R(P) \cong \nabla(\alpha, C, G).$$

Via the above isomorphism, P becomes a quasi-finitely injective cogenerator ∇-comodule. Since ∇ as C-comodule is cofaithfully injective (free C-comodule) it follows that P is a cofaithfully injective C-comodule. Let ϵ denote the composite coalgebra map

$$e_R(P) \longrightarrow \nabla(\alpha, C, G) \xrightarrow{1\otimes\varepsilon} C.$$

Then the cocommutator coalgebra of ϵ is $e_C(P)$ by Lemma 3.4. However, C is a maximal cocommutative quotient coalgebra of ∇, and hence is a maximal cocommutative quotient coalgebra of $e_R(P)$. This implies that $e_C(P) \cong C$. It follows that P as C-comodule is invertible. Thus $P \simeq C$ by [15, 3.23], and

$e_R(P) \cong e_R(C) \cong \nabla(1, C, G)$ by Lemma 2.1. It follows from [4, 6] that α is cohomologous to 1 and we have proved that Φ is injective.

To show that Φ is surjective, (by Prop. 3.6) we may choose each representing element D in $\mathrm{Br}(C/R)$ to be a split Azumaya coalgebra with C being a maximal cocommutative quotient coalgebra of D. For each $\sigma \in G$, there exists a unit $u_\sigma \in U(D^*)$ such that

$$\sigma(\overline{d}) = \sum \langle u_\sigma, d_{(1)} \rangle \overline{d_{(2)}} \langle u_\sigma^{-1}, d_{(3)} \rangle, \quad \forall \overline{d} \in C = \overline{D}.$$

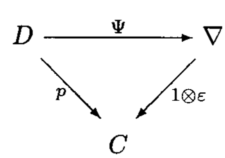
As in the classical algebra case, $\overline{\alpha} \in Z^2(G, C^*)$ defined by $\overline{\alpha}(\sigma, \tau) = u_\sigma u_\tau u_{\sigma\tau}^{-1}$ is a normalized 2-cocycle. Let $\alpha \in Z^2(C, kG^*)$ correspond to $\overline{\alpha}$ in Lemma 3.1. Then we have a crossed coproduct $\nabla(\alpha, C, G)$. Define a map ψ as follows:

$$D \longrightarrow kG^*, \quad \langle \psi(d), \sigma \rangle = \langle d, u_\sigma \rangle.$$

ψ induces a map

$$\Psi\colon D \longrightarrow \nabla(\alpha, C, G), \quad d \mapsto \sum \overline{d_{(1)}} \rtimes_\alpha \psi(d_{(2)}),$$

which satisfies the commutative diagram:



where p is the projection. We verify that Ψ is indeed a coalgebra map. To show

$$\Delta(\Psi(d)) = \sum \Psi(d_{(1)}) \otimes \Psi(d_{(2)}), \quad \forall d \in D,$$

it is sufficient to check that both sides of the above equation have the same action on $kG \otimes kG$ defined by

$$[(c \rtimes p) \otimes (c' \rtimes p')](\sigma \otimes \tau) = c\langle p, \sigma \rangle \otimes c' \langle p', \tau \rangle \in C \otimes C.$$

This works because $C \otimes kG^* \simeq \mathrm{Hom}(kG, C)$. Now for $\sigma, \tau \in G, d \in D$, we have equalities:

$$
\begin{aligned}
&\Delta(\Psi(d))(\sigma\otimes\tau)\\
=&[(\overline{d_{(1)}}\rtimes d_{(2)\langle-1\rangle}\alpha_1(\overline{d_{(3)}})\psi(d_{(4)})_{(1)})\otimes(\overline{d_{(2)\langle 0\rangle}}\rtimes\alpha_2(\overline{d_{(3)}})\psi(d_{(4)})_{(2)})](\sigma\otimes\tau)\\
=&\sum\overline{d_{(1)}}\langle d_{(2)\langle-1\rangle},\sigma\rangle\langle\alpha_1(\overline{d_{(3)}}),\sigma\rangle\langle\psi(d_{(4)})_{(1)},\sigma\rangle\otimes\overline{d_{(2)\langle 0\rangle}}\langle\alpha_2(\overline{d_{(3)}}),\tau\rangle\\
&\langle\psi(d_{(4)})_{(2)},\tau\rangle\\
=&\sum\overline{d_{(1)}}\otimes\overline{d_{(2)\langle 0\rangle}}\langle\alpha(\overline{d_{(3)}}),\sigma\otimes\tau\rangle\langle\psi(d_{(4)}),\sigma\tau\rangle\langle d_{(2)\langle-1\rangle},\sigma\rangle\\
=&\sum\overline{d_{(1)}}\otimes\overline{d_{(2)}\leftharpoonup\sigma}\langle d_{(3)},\overline{\alpha}(\sigma\otimes\tau)\rangle\langle d_{(4)},u_{\sigma\tau}\rangle\\
=&\sum\overline{d_{(1)}}\otimes\overline{d_{(2)}\leftharpoonup\sigma}\langle d_{(3)},u_\sigma u_\tau u_{\sigma\tau}^{-1}\rangle\langle d_{(4)},u_{\sigma\tau}\rangle\\
=&\sum\overline{d_{(1)}}\otimes\overline{d_{(2)}\leftharpoonup\sigma}\langle d_{(3)},u_\sigma\rangle\langle d_{(4)},u_\tau\rangle\\
=&\sum\overline{d_{(1)}}\otimes\overline{d_{(3)}}\langle d_{(2)},u_\sigma\rangle\langle d_{(4)},u_{\sigma^{-1}}\rangle\langle d_{(5)},u_\sigma\rangle\langle d_{(6)},u_\tau\rangle\\
=&\sum\overline{d_{(1)}}\langle d_{(2)},u_\sigma\rangle\otimes\overline{d_{(3)}}\langle d_{(4)},u_\tau\rangle\\
=&\sum\Psi(d_{(1)})(\sigma)\otimes\Psi(d_{(2)})(\tau);
\end{aligned}
$$

Ψ is surjective because $\Psi(D)=\nabla\square R'$ for some subcoalgebra R' of R and Ψ is an R-coalgebra map which forces $R=\epsilon(D)=\epsilon\Psi(D)=\epsilon(\nabla\square R')=R'$. Since $\nabla(\alpha,C,G)$ is an Azumaya R-coalgebra, we have a coalgebra isomorphism:

$$(\Psi\square\Psi^d)\Delta\colon D\longrightarrow\nabla\square Z_\nabla(D),$$

and $Z=Z_\nabla(D)$ is an Azumaya R-coalgebra by [15, 3.21]. Since (Z,Ψ^d) cocommutes with (∇,Ψ), it cocommutes with (C,p) by the foregoing commutative diagram. But $(C,p)=(Z_D(C),p^d)$ by our choice. There exists a unique (surjective) coalgebra map $\eta\colon C\longrightarrow Z$ which completes the following commutative diagram:

$$
\begin{array}{ccc}
D & \xrightarrow{\ \Psi\ } & \nabla\\
{\scriptstyle\Psi^d}\downarrow & \searrow{\scriptstyle p} & \downarrow{\scriptstyle 1\otimes\varepsilon}\\
Z & \xleftarrow{\ \eta\ } & C.
\end{array}
$$

We obtain a surjective coalgebra map $\nu=\eta(1\otimes\varepsilon)\colon\nabla\longrightarrow Z$. This coalgebra map cocommutes with the identity map 1_∇. Indeed, since Ψ is surjective, each element δ of ∇ may be written as an image $\Psi(d)$ for some $d \in D$.

$$\begin{aligned}\sum \delta_{(1)} \otimes \nu(\delta_{(2)} &= \sum \Psi(d_{(1)}) \otimes \nu\Psi(d_{(2)}) \\ &= \sum \Psi(d_{(1)}) \otimes \Psi^d(d_{(2)}) \\ &= \sum \Psi(d_{(2)}) \otimes \Psi^d(d_{(1)}) \\ &= \sum \delta_{(2)} \otimes \nu(\delta_{(1)}).\end{aligned}$$

Note that ν is surjective. It follows that Z is a cocommutative coalgebra, and the cocenter of Z is itself. On the other hand, Z is an Azumaya R-coalgebra. So the cocenter is R. Therefore, $Z = R$, and Ψ is an isomorphism. ∎

COROLLARY 3.9: *Let C/R be a kG-Galois coextension. Then the transpose map $[D] \longrightarrow [D^*]$ defines an isomorphism from* $\mathrm{Br}(C/R)$ *to* $\mathrm{Br}(C^*/R^*)$.

Proof: Follows from Lemma 3.1, Theorem 3.8, and the classical crossed product theorem for the Galois extension C^*/R^*. ∎

Now we are able to show that the Brauer group of an irreducible coreflexive coalgebra R is the union of the relative Brauer groups. Precisely, any Azumaya R-coalgebra is split by a Galois coextension of R. This gives an affirmative answer to the question proposed in [15, p. 568]. Recall that a coalgebra D is **coreflexive** if $C = (C^*)^0$, where A^0 of an algebra A is defined as the maximal coalgebra in A^*.

THEOREM 3.10: *Let R be an irreducible coreflexive cocommutative coalgebra. Then* $\mathrm{Br}(R) = \mathrm{Br}(R^*)$.

Proof: We know that the transpose map in Cor.3.9 gives a monomorphism from $\mathrm{Br}(R)$ to $\mathrm{Br}(R^*)$ cf. [15, 4.12]. As shown in [15, 4.13] R^* is a completely local algebra. It is well-known that the Brauer group of a completely local ring (or algebra) is equal to the Brauer group of the residual field and hence is equal to the union of the relative Brauer groups. Let $\mathrm{Br}(R^*) = \bigcup \mathrm{Br}(A/R^*)$, where A runs over all the Galois extensions of R^*. If we can show that any Galois extension A of R^* is a dual of some coalgebra C which is a Galois coextension of R, then applying Corollary 3.9 we obtain that $\mathrm{Br}(R^*)$ is full of $\mathrm{Br}(R^*)$.

Let A/R^* be a Galois extension with respect to a finite group G_A. Let ω be the maximal ideal of the local algebra R^* such that $R^*/\omega = k$. Since A/R^* is a finite Galois extension and R is local, A is a finitely generated free R-module. This yields that $A/\omega A \cong k \otimes_{R^*} A$ is a finite dimensional extension of k. In fact, $\overline{A} \equiv A/\omega A$ is a Galois extension of k with respect to the same group G_A, where G acts on $\overline{A}$ via the acting of G on A, that is, $g(a \otimes u) = g(a) \otimes u$, $g \in G$, $a \in A$, $u \in k$. The proof of this fact is easy. For instance, the canonical Galois isomorphism:

$$\overline{A} \otimes \overline{A} \longrightarrow \overline{A} \otimes kG_A^*, \quad \bar{a} \otimes \bar{b} \mapsto \sum \bar{a}\bar{b}_{(0)} \otimes \bar{b}_{(1)}$$

(here $\overline{A}$ is a right kG_A^*-comodule algebra) may be obtained in the following way:

$$\overline{A} \otimes \overline{A} \cong (k \otimes_{R^*} A) \otimes (k \otimes_{R^*} A \cong k \otimes_{R^*} (A \otimes_{R^*} A) \cong k \otimes_{R^*} (A \otimes kG_A^*) = \overline{A} \otimes kG_A^*.$$

Since $\overline{A}$ is finite dimensional, we may consider its dual coalgebra $\overline{A}^*$, denoted by D. It is obvious that D is a kG_A-Galois coextension of k. It follows from (7) that $R \otimes D$ is a kG_A^* (or $k(1 \times G_A^*)$)-Galois coextension of R. Now let C be $R \otimes D$. We have that

$$C^* = (R \otimes D)^* \cong R^* \otimes D^* \cong R^* \otimes \overline{A} = R^* \otimes (k \otimes_{R^*} A) \cong A,$$

where the first isomorphism holds because D is finite dimensional. Thus we proved that the coalgebra C is the desired coalgebra, and the proof is done. ∎

COROLLARY 3.11: *Let R be an irreducible coreflexive cocommutative coalgebra. Then every Azumaya R-coalgebra is split by a Galois coextension of R, and the Brauer group* $\mathrm{Br}(R)$ *may be interpreted as the union of the second Galois cohomology groups.*

Proof: Follows from Th. 3.8, Cor. 3.9 and Th. 3.10. ∎

Note that in general the transpose map $(*)$ in Cor. 3.9 does not induce an isomorphism from $\mathrm{Br}(R)$ to $\mathrm{Br}(R^*)$, and the classical splitting theorem for the Brauer group of a field fails in general.

Example 3.12: Let R be a group-like coalgebra QS, where Q is the rational field and S is an infinite set. We know that $\mathrm{Br}(R) = \prod_S \mathrm{Br}(Q)$ cf. [15]. Let K be a (usual) Galois extension of Q with Galois group $G = \mathrm{Gal}(K/Q)$. Then K is finte dimensional over Q. It follows that K^*/Q is a G-Galois coextension, and $\bigoplus_{s\in S} K_s^*/QS$, $K_s^* = K^*$, is a G-Galois coextension (see the proof of Prop. 2.5). Moreover, all the Galois coextensions of QS arise in this way. It is well-known that $\mathrm{Br}(Q) = \bigcup_K \mathrm{Br}(K/Q)$, K runs over the Galois extensions of Q. Thus $\mathrm{Br}(R) = \prod_S \mathrm{Br}(Q) = \prod_S(\bigcup_K \mathrm{Br}(K/Q))$, which is not a torsion group. On the other hand, the union of the relative Brauer groups $\mathrm{Br}(C/R)$, C/R a Galois coextension, is torsion by Theorem 3.8. So $\mathrm{Br}(QS)$ cannot be the union of the relative Brauer groups.

To complete this paper let us give an example of the crossed coproducts as Azumaya coalgebras.

Example 3.13: Let R be the field of real numbers. Let C be a coalgebra over R generated by two elements i, j as follows:

$$\epsilon(i) = 1, \ \epsilon(j) = 0,$$
$$\Delta(i) = i \otimes i - j \otimes j, \qquad \text{and}$$
$$\Delta(j) = i \otimes j + j \otimes i.$$

Let $G = \{1, e\}$ be the cyclic group of order 2. Define a G-action on C given by

$$i \leftharpoonup e = i, \quad j \leftharpoonup e = -j.$$

Then C is a right RG-module coalgebra, equivalently a left RG^*-comodule coalgebra. Let p_1, p_e be the orthogonal basis of RG^*. Then the left comodule structure of C is given by

$$\rho(i) = (p_1 + p_e) \otimes i = 1 \otimes i, \quad \rho(j) = (p_1 - p_e) \otimes j.$$

It is straightforward to check that C/R is an RG-Galois coextension. Now we construct a nontrivial 2-cocycle σ of $Z^2(C, RG^*)$ as follows:

$$\sigma(i) = p_1 \otimes p_1 + p_1 \otimes p_e + p_e \otimes p_1 - p_e \otimes p_e, \quad \sigma(j) = 0.$$

It is routine verification that σ satisfies the cocyle conditions:

(1) $[(1 \otimes \sigma)\rho] * [(1 \otimes \Delta)\sigma] = [(\Delta \otimes 1)\sigma] * [f_3\sigma]$, where $f_3(x \otimes y) = x \otimes y \otimes 1$, $x, y \in RG^*$,

(2) $\sum \epsilon_{RG^*}(\sigma(c)_1)\sigma(c)_2 = \epsilon_C(c)1_{RG^*} = \sum \epsilon_{RG^*}(\sigma(c)_2)\sigma(c)_1$ for all $c \in C$.

Thus the crossed coproduct $D = C \rtimes_\sigma RG^*$ is an R-coalgebra split by C cf. (9). We show that D is not an elementary Azumaya R-coalgebra. Let Q_4 be an

R-coalgebra generated by four elements a, b, c and d as follows:

$$\Delta(a) = a \otimes a - b \otimes b - c \otimes c - d \otimes d,$$
$$\Delta(b) = a \otimes b + b \otimes a + c \otimes d - d \otimes c,$$
$$\Delta(c) = a \otimes c + c \otimes a + d \otimes b - b \otimes d,$$
$$\Delta(d) = a \otimes d + c \otimes d + b \otimes c - c \otimes b, \quad \text{and}$$
$$\epsilon(a) = 1, \epsilon(b) = \epsilon(c) = \epsilon(d) = 0.$$

Define a map θ from $D = \nabla(\sigma, C, G)$ to Q_4 given by

$$\theta(i \rtimes p_1) = a, \quad \theta(i \rtimes p_e) = b, \quad \theta(j \rtimes p_1) = c, \quad \text{and} \quad \theta(j \rtimes p_e) = d.$$

Now a straightforward computation shows that θ is an R-coalgebra isomorphism. However, Q_4^* is the quaternion algebra $R\langle -1, -1\rangle$ which is a nontrivial Azumaya R-algebra. It follows that D is nontrivial. It is well-known that $\mathrm{Br}(C^*/R) = \mathbf{Z_2}$ with the nontrivial element $[R\langle -1, -1\rangle]$, where C^* is the complex field. It follows from Cor. 3.9 that $\mathrm{Br}(C/R) \cong H^2(C, RG^*) = \mathbf{Z_2}$.

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